# Value-at-Risk FOR QUADRATIC PORTFOLIO'S WITH GENERALIZED LAPLACE DISTRIBUTED RISK FACTORS

**Jules SADEFO KAMDEM** [1], *Laboratoire de Mathématique (UMR 6056 - CNRS) Reims, France*, sadefo@univ-reims.fr

*Key words*: Asymptotic analysis, $\Delta$-$\Gamma$ VaR, Risk Management ...

This paper is concerned with the efficient numerical computation of static Value-at-Risk (VaR) for portfolios of assets depending quadratically on a large number of risk factors $\mathbb{X}_t = (X_{1,t}, \cdots, X_{n+1,t})$ ($t$ representing time), under the assumption that $\mathbb{X}_t$ follows a Generalized Laplace Distribution or GLD. Our approach is designed to supplement the usual Monte-Carlo techniques, by providing an asymptotic formula for the quadratic portfolio's cumulative distribution function, together with explicit error-estimates. The basic philosophy is the same as in Brummelhuis, Cordoba, Quintanilla and Seco [1], where such an asymptotic formula was derived in the case of normally distributed risk factors. Here the result of [1] will be extended to a class of non-Gaussian $\mathbb{X}_t$'s, and even slightly improved upon in the normal case). More importantly, the asymptotic formula will be supplemented with estimates for the error-term, which were lacking in [1]. This will allow us to establish a rigorous interval in which the true quadratic VaR will lie, rather than just give an approximation which is asymptotically exact when the VaR confidence parameter tends to 1.

The typical way in which quadratic portfolios arise in practice are as a $\Gamma - \Delta$ approximations of more complicated portfolios with some non-linear value function $\Pi(X_{1,t}, \cdots, X_{n+1,t}, t)$. We will make the additional assumption that $\Pi$ is *delta-hedged at* $t = 0$. The restriction to $\Delta$-hedged portfolios is mainly made for computational simplicity, but note that these include the in practice very important class of hedging portfolios made up of derivatives and their underlying. In such a case, letting $S_{j,t}$ be the time-$t$ price of the $j$-th underlying asset, we would typically take the log-return $X_{j,t} = \log(S_{j,t}/S_{j,0})$ as the $j$-th risk factor. The numerical example we consider at the end of the paper will be of this kind. A further assumption we will make is that $\mathbb{X}_t$ has zero mean, which in practice will be approximately satisfied on small time-scales $t$. We stress, however, that all results of this paper can be extended to general, non-hedged, quadratic portfolios with $\mathbb{X}_t$ having a non-zero mean. But, we decided to postpone the more general case to a future paper, and first test our approach on the $\Delta$-hedged case.

Why would one want to derive explicit analytic approximations to a portfolio's VaR when simple Monte Carlo will in principle compute this with any given precision? There are in fact a number of good reasons for wanting to do so. First of all, Monte Carlo, even when combined with various variance reduction and/or importance sampling techniques, can be notoriously slow for large portfolios. By contrast, explicit analytical expressions can in general be computed almost instantaneously, and would allow for real-time VaR evaluation. Doing this by Monte Carlo would involve massive computations, and therefore likely to be unfeasible in practice. On the other hand, explicit analytical expressions, even if approximate or providing bounds only, will easily permit such an analysis.

To begin describing our main results, consider a portfolio with non-linear *Profit and Loss (or P & L) function*[2] $\Pi = \Pi(x_1, \cdots, x_{n+1}, t)$ over the time-interval $[0, t]$. In particular, $\Pi(0,0) = 0$, assuming (without loss of generality) that $\mathbb{X}_0 = 0$. We suppose moreover that $\nabla\Pi(0,0) = 0$. When $t > 0$ is some small fixed later time (typically of the order of 1 day, or 1/252 in the natural unit of one financial year), To compute VaR, we need to know the P&L's cdf

$$F_{\Pi_t}(x) = \mathbb{P}\,(\,\Pi(\mathbb{X}_t, t) < x\,)\,, \tag{1}$$

$\mathbb{P}$ standing for the objective probability. Since the distribution function (1) is in general impossible to evaluate analytically, and, for big $n$ and complicate $\Pi(x_1, \cdots, x_{n+1}, t)$, time-consuming to compute numerically by Monte Carlo, one usually performs a preliminary quadratic approximation:

$$\begin{aligned} \Pi(\mathbb{X}) &\simeq \Theta\, t + \frac{1}{2}\mathbb{X}_t \Gamma \mathbb{X}_t^t \\ &= \Theta\, t + \frac{1}{2}\sum_{j,k} \Gamma_{ij} X_i X_j, \end{aligned} \tag{2}$$

where there is no linear term since $\Pi$ is assumed to be $\Delta$-hedged. Here, and below, we will use the following notational conventions for vectors and matrices: $x = (x_1, \cdots, x_{n+1})$ and $\mathbb{X} = (X_1, \cdots, X_{n+1})$ will designate row vectors, and their transposes $x^t$, $\mathbb{X}^t$ will therefore be column vectors, on which matrices like $\Gamma = (\Gamma_{ij})_{i,j}$ act by left multiplication As of now we assume that $\mathbb{X}_t$ has probability density of the form:

$$f_{\mathbb{X}_t}(x) = \frac{C_{\alpha,n+1}}{\sqrt{\det\,(\mathbb{V}(t))}} \exp\left(-c_{\alpha,n+1}(x\mathbb{V}(t)x^t)^{\alpha/2}\right), \tag{3}$$

where $\alpha > 0$ and where $\mathbb{V}(t)$ is a positive definite matrix; $\mathbb{V}(t)$ will precisely be $\mathbb{X}_t$'s variance-covariance matrix, provided we choose the normalization constants $C_{\alpha,n+1}$ and $c_{\alpha,n+1}$.

The Value-at-Risk at (risk-managerial) confidence level $1 - p$ is defined by

$$\mathrm{VaR}_p^{\Pi_t} = \sup\{V : F_{\Pi_t}(-V) \geq p\}. \tag{4}$$

We will assume that a reasonable approximation to $\mathrm{VaR}_p$ will be given by the quadratic or $\Gamma$-Value-at-Risk, $\mathrm{VaR}_p^{\Gamma_t}$, defined as in (4), but with $F_{\Pi_t}$ replaced by

$$F_{\Gamma_t}(-V) = \mathbb{P}(\,\Theta t + \frac{1}{2}\mathbb{X}_t\,\Gamma\,\mathbb{X}_t^t \leq -V\,). \tag{5}$$

In our case, the distribution function $F_\Gamma$ will be strictly increasing, so the definition of $\Gamma$-VaR simplifies to $F_{\Gamma_t}^{-1}(p)$. In [5] we have proved that

$$\mathrm{VaR}_p^{\Pi_t}/\mathrm{VaR}_p^{\Gamma_t} \to 1, \quad t \to 0,$$

with an error which is $O(\sqrt{t})$. Also observe that if for example $\Pi(x,t) \geq \Theta_t + \frac{1}{2}x\Gamma x^t$ for all $x$, then of course $\mathrm{VaR}_p^{\Pi_t} \leq \mathrm{VaR}_p^{\Gamma_t}$, and similarly with all inequality signs reversed. From now on, we will

[2]we use the P & L rather than the value function; this is of course just a question of normalization

take $t$ sufficiently small but fixed, and make no distinction any more between $\mathrm{VaR}_p^{\Pi_t}$ and $\mathrm{VaR}_p^{\Gamma_t}$, that is, we will effectively suppose that $\Pi(x,t)$ is a quadratic $\Delta$-hedged portfolio.

Our main task will then be to compute $F_\Gamma(-V)$, or more precisely its inverse. This is still a non-trivial problem if we are looking for an analytic solution (which we are, for though Monte Carlo works faster for quadratic portfolios, it will still be slow if the portfolio is big). Our strategy will be to approximate $F_\Gamma(-V)$ for large values of $V$ by an explicit analytic expression, with explicit error bounds. This will then allow an approximate inversion.

The main theorem can be used as follows to solve our initial problem of finding good approximations and bounds for $\mathrm{VaR}_p^\Gamma$. Let us define the *principal component* $\Gamma$*-VaR* of our quadratic portfolio as the unique solution $V = \mathrm{VaR}_p^{\Gamma,\mathrm{pc}}$ of the equation

$$F_{\Gamma,\mathrm{pc}}\left(c_{\alpha,n+1}^{1/\alpha}\sqrt{2(V+\Theta)}\right) = p. \tag{6}$$

Theorem **??** then suggests, as a first approximation,

$$\mathrm{VaR}_p^\Gamma \simeq \mathrm{VaR}_p^{\Gamma,\mathrm{pc}},$$

a relation which is asymptotically exact as $p \to 0$. For a given small but non-zero $p > 0$ this is, as it stands, just an uncontrolled approximation, but we can use the error bounds of theorem **??** to determine a rigorous interval in which $\mathrm{VaR}_p^\Gamma$ must lie. For a given $p \in (0,1)$, let $R_L = R_L(p)$ and $R_U = R_U(p)$ solve, respectively:

$$F_{\Gamma,\mathrm{pc}}(R_L) - \mathcal{E}_L(R_L) = p, \tag{7}$$

and

$$F_{\Gamma,\mathrm{pc}}(R_L) + \mathcal{E}_U(R_U) = p. \tag{8}$$

Put

$$V_j(p) := \frac{1}{2}c_{\alpha,n+1}^{-2/\alpha}R_j(p)^2 - \Theta, \quad j = L, U. \tag{9}$$

Since the lower bound (**??**) holds for all $R > 0$, we will always have that $V_L(p) \leq \mathrm{VaR}_p^\Gamma$. On the other hand, $\mathrm{VaR}_p^\Gamma \leq V_U(p)$ will only hold once we know that $c_{n+1,\alpha}2^{\alpha/2}\left(\mathrm{VaR}_p^\Gamma + \Theta\right)^{\alpha/2} \geq \lambda_0$. This will certainly be satisfied if we choose:

$$\lambda_0 = R_L(p)^\alpha \tag{10}$$

Summarizing, we then have the following estimate on quadratic VaR:

**Corollary 0.1** *For a given choice of parameters $p, a, \gamma \in (0,1)$ let $\lambda_0 = R_L(p)^\alpha$, where $R_L(p)$ is the solution of (7). Furthermore, for given $\varepsilon \in (0,1)$, let $R_U(p)$ be the solution of (8)*[3]*. Let $V_L(p), V_U(p)$ be defined by (9). Then:*

$$\mathrm{VaR}_p^\Gamma \in [V_L(p), V_U(p)].$$

[3] that is, with this choice of parameters in the expressions for $\mathcal{E}_L(R), \mathcal{E}_U(R)$

## References

[1] Brummelhuis, R., Córdoba, Quintanilla, M. and Seco, L. (2002) *Principal component value at risk, Math. Finance* **12** *(1), 1 - 21.*
[2] Cardenas, J., Fruchard, E., Koehler, E., Michel, C. and Thomazeau (1998) *VAR:One step beyond, Risk* **10**, *72 - 75.*
[3] Glasserman, P., Heidelberger, P. and Shahabuddin, P. (2002):: *Portfolio value-at-risk with heavy-tailed risk factors, Math. Finance* **12** *(3), 239 - 269.*
[4] Rouvinez, C (1997): *Going Greek with VaR, Risk* **10**, *57-65.*
[5] Sadefo Kamdem, J (2004): *Méthodes analytiques pour le Risque des Portefeuilles Financiers, PhD thesis, Université de Reims.* , . . .